INTEGRAL GEOMETRY IN CAYLEY SPACES

by

ARTHUR HOWARD SIMONSON

B. A., Phillips University, 1964

A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Mathematics

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1967

Approved by:

Cheng-Jung Hsu

Major Professor

TABLE OF CONTENTS

PRELIMINARIES

Integral geometry emerged from probability problems involving geometric figures, one of the classical examples being G. L. L. Comte de Buffon's needle problem (1760). The title "Integral Geometry" was given later in these investigations by Wilhelm Blaschke in a series of works by his school and in his "Vorlesungen über Integralgeometrie" (1936).

The question raised by early studies in geometric probability was reduced to the problem of how one can find a "measure" for sets of geometrical objects, such as points, lines, planes, conics, etc., so that these measures are invariant under a given group of transformations. This search has been enlarged and systemized by studies in integral geometry: once an invariant measure is found, geometrical consequences for the figures of the space in which the group operates are derived. Thus integral geometry, while it maintains a kinship to differential geometry in some of its methods, differs in that its results are "global" while those of differential geometry are "local".

Treatment of the case of elliptic spaces first arose in G. Herglotz's lectures on geometrical probability. Subsequently these results were extended in the realm of integral geometry. One curious result was found. Often formulas arising from the systematic treatment of something in the elliptic case were the same as those obtained in the Euclidean case! Here lies the clue to the power of investigation in elliptic spaces. Elliptic spaces are compact. They offer a convenient principle of duality. Using these facts, many new integral formulas, which take

on infinite values in the Euclidean case, may be obtained.

Two supplementary statements should be made at this point. First, since elliptic geometry and spherical geometry are locally the same, the results pertaining to elliptic geometry also apply, with slight modification, to the geometry on the sphere. Second, using the same procedures followed for the treatment of the elliptic case, the hyperbolic case can also be handled.

Elliptic geometry is related to the transformation group called the group of Cayley. This is the group (in n-dimensional projective space) of all projectivities which leave invariant a given quadratic form. All elliptic spaces considered herein will be of curvature 1. This being the case, the distance d between two points x and y is defined by $\cos d = (xy)$; d also represents the angle between the two corresponding polar elements. In other words, in the plane, d is the angle between two lines; in space, d is the angle between two planes, and so on. The following general results, which will be used in the sequel, are for n-dimensional elliptic space.

<u>(I-1)</u> The relative components for the group of Cayley are given by the linear differential forms $\omega_{ik}(i, k = 0, 1, \ldots, n)$ where $\omega_{ii} = 0$

$$\omega_{ik} = -\omega_{ki} = (A_k dA_i) = -(dA_k A_i) ,$$

(parentheses denote the scalar product) and the $A_0, A_1, \ldots, A_n$ are vertices of a self-conjugate figure; that is,

$$(A_i A_k) = \begin{cases} 1 & i = k \\ 0 & i \neq k. \end{cases}$$

(I-2) The ω_{ik} are given by an infinitesimal transformation and are defined by

$$dA_i = \sum_{k=0}^{n} \omega_{ik} A_k \ .$$

(I-3) The Maurer-Cartan equations of structures are

$$\omega_{ik}{}' = \sum_{j=0}^{n} \left[\omega_{ij}\omega_{jk}\right] \ .$$

(The brackets denote the exterior product of the differential forms.)

(I-4) A necessary and sufficient condition for

$$\int \left[\omega_1 \ \omega_2 \ . \ . \ . \ \omega_h\right]$$

to be a measure for geometric elements H, is that

$$\left[\omega_1 \ \omega_2 \ . \ . \ . \ \omega_h\right]' = 0.$$

Then $\left[\omega_1 \ \omega_2 \ . \ . \ . \ \omega_h\right]$ is denoted by dH. The $\omega_1, \omega_2, \ldots, \omega_h$ are determined in the following manner: they are those independent relative components of the group of Cayley transformations which must be zero in order for the geometric element H to be transformed into itself. Related to these facts is the notion that for the group of Cayley, the kinematic density, which is the product of all independent relative components, always exists.

(I-5) Three fundamental invariant properties of the kinematic density need to be mentioned at this point. Kinematic density is invariant first under a motion. The second property is invariance of choice; that is, the kinematic density does not change if the coordinate system is changed. Third is invariance under inversion: if the original fixed axes are regarded as mobile and the original mobile axes are regarded as fixed, the kinematic density will remain the same.

ELLIPTIC SPACE OF 2-DIMENSIONS

In this section, some results of integral geometry in elliptic 2-space will be considered. Later sections will extend these to 3-space and n-space. Throughout this particular discussion A_0, A_1, and A_2 will represent vertices of a self-conjugate triangle as defined in (I-1).

Theorem II-1. The density of the point A_0 is

$$dA_0 = \left[\omega_{01}\omega_{02}\right] = (dA_0A_1)(dA_0A_2)$$

Proof. From (I-2)

$$\begin{aligned} dA_0 &= \omega_{00}A_0 + \omega_{01}A_1 + \omega_{02}A_2 \\ &= \omega_{01}A_1 + \omega_{02}A_2 \end{aligned}$$

by (I-1). A_0 is fixed by the infinitesimal transformation so that $dA_0 = 0$. Thus $\omega_{01} = 0$, $\omega_{02} = 0$, so that, considering (I-4), the theorem is proved provided $\left[\omega_{01}\omega_{02}\right]' = 0$. Now

$$\left[\omega_{01}\omega_{02}\right]' = \left[\omega_{01}'\omega_{02}\right] - \left[\omega_{01}\omega_{02}'\right]$$

However, (I-3) gives that

$$\omega_{01}{}' = [\omega_{00}\omega_{01}] + [\omega_{01}\omega_{11}] + [\omega_{02}\omega_{21}]$$
$$= [\omega_{02}\omega_{21}]$$

by (I-1); furthermore,

$$\omega_{02}{}' = [\omega_{00}\omega_{02}] + [\omega_{01}\omega_{12}] + [\omega_{02}\omega_{22}]$$
$$= [\omega_{01}\omega_{12}]$$

by (I-1) also. Thus

$$[\omega_{01}\omega_{02}]' = [\omega_{02}\omega_{21}\omega_{02}] - [\omega_{01}\omega_{01}\omega_{12}]$$
$$= 0.$$

Q. E. D.

The density about a point is sometimes referred to as the area element described by that point.

Theorem II-2. If A_1 and A_2 determine a line G, then the density of G is

$$dG = [\omega_{10}\omega_{20}] = (dA_0A_1)(dA_0A_2).$$

Proof. Consider a subgroup of infinitesimal transformations which fix G; such a transformation must take

$$A_1 \longrightarrow A_1 + dA_1$$

$$A_2 \longrightarrow A_2 + dA_2$$

where $A_1 + dA_1$ and $A_2 + dA_2$ are points on the line G. Thus dA_1 and dA_2 must be linear combinations of A_1 and A_2. From

(I-1) and (I-2)

$$dA_1 = \omega_{10}A_0 + \omega_{11}A_1 + \omega_{12}A_2 = \omega_{10}A_0 + \omega_{12}A_2$$

and

$$dA_2 = \omega_{20}A_0 + \omega_{21}A_1 + \omega_{22}A_2 = \omega_{20}A_0 + \omega_{21}A_1 \ .$$

Thus from considering the linear combinations mentioned above, $\omega_{10} = 0$ and $\omega_{20} = 0$, so that from (I-4)

$$dG = \left[\omega_{10}\omega_{20}\right] ,$$

if it can be shown that $\left[\omega_{10}\omega_{20}\right]' = 0$. The proof of this follows from (I-3) as in Theorem II-1.

It is of interest to note that $dA_0 = dG$. This result is expected, however, since the dual of the point A_0 is the line A_1A_2. Q. E. D.

Theorem II-3. The kinematic density in the plane is

$$dK = \left[\omega_{01}\omega_{02}\omega_{12}\right] = (dA_0A_1)(dA_0A_2)(dA_1A_2).$$

Proof. This follows from (I-4) since among

$$\begin{matrix} \omega_{00} & \omega_{01} & \omega_{02} \\ \omega_{10} & \omega_{11} & \omega_{12} \\ \omega_{20} & \omega_{21} & \omega_{22} \end{matrix}$$

$\omega_{00} = \omega_{11} = \omega_{22} = 0$ and $\omega_{01} = -\omega_{10}$, $\omega_{02} = -\omega_{20}$, $\omega_{12} = -\omega_{21}$ so that three independent pfaff forms are ω_{01}, ω_{02}, ω_{21}. Q.E.D.

Some other results from 2-space will be mentioned in the following section where they will appear following the corresponding result and proof for the case of 3-space.

ELLIPTIC SPACE OF 3-DIMENSIONS

The natural path at this point is to consider the possibilities of the previous section in 3-space. It should be noted that the first four theorems which follow, very closely parallel the three of the preceding section.

Theorem III-1. Let $A_0A_1A_2A_3$ be a self-conjugate tetrahedron as defined in (I-1). The density of the point A_0 is

$$dA_0 = \left[\omega_{01}\omega_{02}\omega_{03}\right] = (dA_0A_1)(dA_0A_2)(dA_0A_3).$$

Proof: (I-2) gives

$$dA_0 = \omega_{00}A_0 + \omega_{01}A_1 + \omega_{02}A_2 + \omega_{03}A_3 \ .$$

Moreover, $\omega_{00} = 0$ and since A_0 is fixed, $dA_0 = 0$. Thus according to (I-4), the result desired follows if $\left[\omega_{01}\omega_{02}\omega_{03}\right]' = 0$. This verification is routine when use is made of the Maurer-Cartan equations (I-3). Q. E. D.

Theorem III-2. If $A_0A_1A_2A_3$ is a self-conjugate tetrahedron and if $G = A_0A_1$ is a line, then the density of the line G is

$$dG = \left[\omega_{02}\omega_{03}\omega_{12}\omega_{13}\right] = (dA_0A_2)(dA_0A_3)(dA_1A_2)(dA_1A_3).$$

Proof. Just as in the proof of Theorem II-2, consider the transformation that takes

$$A_0 \longrightarrow A_0 + dA_0$$

$$A_1 \longrightarrow A_1 + dA_1$$

and leaves G fixed; $A_0 + dA_0$ and $A_1 + dA_1$ must be points on G so that dA_0 and dA_1 are linear combinations of A_0 and A_1. From (I-2)

$$dA_0 = \omega_{01}A_1 + \omega_{02}A_2 + \omega_{03}A_3$$

$$dA_1 = \omega_{10}A_0 + \omega_{12}A_2 + \omega_{13}A_3 ;$$

thus $\omega_{02} = 0$, $\omega_{03} = 0$, $\omega_{12} = 0$, $\omega_{13} = 0$. The theorem follows if $\left[\omega_{02}\omega_{03}\omega_{12}\omega_{13}\right]' = 0$. This is again routine. Q. E. D.

Theorem III-3. Let $A_0A_1A_2A_3$ be a self-conjugate tetrahedron; if $E = A_1A_2A_3$ is a plane, then the density of E is

$$dE = \left[\omega_{10}\omega_{20}\omega_{30}\right] = (dA_1A_0)(dA_2A_0)(dA_3A_0).$$

Proof: Consider the transformation that takes

$$A_1 \longrightarrow A_1 + dA_1$$

$$A_2 \longrightarrow A_2 + dA_2$$

$$A_3 \longrightarrow A_3 + dA_3;$$

E must be fixed by this transformation so that, as before, dA_1, dA_2, and dA_3 must be linear combinations of A_1, A_2, and A_3. Therefore

$$dA_1 = \omega_{10}A_0 + \omega_{12}A_2 + \omega_{13}A_3$$

$$dA_2 = \omega_{20}A_0 + \omega_{21}A_1 + \omega_{23}A_3$$

$$dA_3 = \omega_{30}A_0 + \omega_{31}A_1 + \omega_{32}A_2$$

gives that $\omega_{10} = 0$, $\omega_{20} = 0$, and $\omega_{30} = 0$. The rest of the proof follows that of Theorem II-2. Q. E. D.

Here in 3-space duality also appears. Note that (except for sign) the density of a point and of a plane are the same. Densities are always considered to be positive.

Theorem III-4. The kinematic density in 3-space is

$$dK = \left[\omega_{01}\omega_{02}\omega_{03}\omega_{12}\omega_{13}\omega_{23}\right]$$

$$= (dA_0A_1)(dA_0A_2)(dA_0A_3)(dA_1A_2)(dA_1A_3)(dA_2A_3)$$

where $A_0A_1A_2A_3$ is a self-conjugate tetrahedron.

Proof. The independent pfaff forms among

$$\begin{array}{cccc} \omega_{00} & \omega_{01} & \omega_{02} & \omega_{03} \\ \omega_{10} & \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{20} & \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{30} & \omega_{31} & \omega_{32} & \omega_{33} \end{array}$$

are by (I-1): ω_{01}, ω_{02}, ω_{03}, ω_{12}, ω_{13}, ω_{23}. The desired result follows from (I-4). Q. E. D.

In elliptic 3-space some formulas can be derived which have analogies in 2-space. The situation that occurs when planes intersect a curve will be considered first; in preparation for this an expression will be derived for dE in which one of the three points determining E moves in a certain direction.

Choose $A_0A_1A_2A_3$ to be a self-conjugate tetrahedron. For purposes of symmetry, the plane E is determined by $A_0A_2A_3$. A_1 is the pole of E because $A_0A_1A_2A_3$ is a self-conjugate tetrahedron. By Theorem III-3,

$$dE = (dA_0A_1)(dA_2A_1)(dA_3A_1).$$

Now a point A on the plane $A_1A_2A_3$ is chosen and its coordinates normalized. Thus A must satisfy

$$A = a_1A_1 + a_2A_2 + a_3A_3 \quad (a_1^2 + a_2^2 + a_3^2 = 1). \qquad (1)$$

Then the equation $(AX) = 0$ represents the set of all points conjugate to A, or, in other words, a plane. This plane passes through A_0:

$$(AX) = a_1(A_1X) + a_2(A_2X) + a_3(A_3X) = 0$$

and $X = A_0$ satisfies this equation because $A_0A_1A_2A_3$ is a self-conjugate tetrahedron. Denote by B_k the intersection of the plane $(AX) = 0$ with the line A_iA_j where i, j, k is a cyclic permutation of 1, 2, 3 $[(i, j, k) = (1, 2, 3)]$. B_k is on the line joining A_i and A_j so that

$$B_k = c_1A_i + c_2A_j \;;$$

moreover, B_k is on the plane $(AX) = 0$, giving

$$(AB_k) = \sum_{m=1}^{3} a_mA_m(c_1A_i + c_2A_j)$$

$$= \sum_{m=1}^{3} a_m c_1 (A_m A_i) + a_m c_2 (A_m A_j)$$

$$= a_i c_1 + a_j c_2 = 0.$$

It is advantageous to have the coordinates for B_k normalized so that c_1 and c_2 will also satisfy

$$c_1^2 + c_2^2 = 1.$$

Therefore, because

$$a_i^2 + a_j^2 + a_k^2 = 1,$$

or

$$\frac{a_i^2}{1 - a_k^2} + \frac{a_j^2}{1 - a_k^2} = 1,$$

the combined equations give as a solution

$$B_k = \frac{a_j}{\sqrt{1 - a_k^2}} A_i - \frac{a_i}{\sqrt{1 - a_k^2}} A_j, \quad (i, j, k) = (1, 2, 3). \tag{2}$$

These relations, when written out for $k = 1, 2, 3$, give

$$a_1 \sqrt{1 - a_1^2}\, B_1 = a_1(a_3 A_2 - a_2 A_3) \tag{3-a}$$

$$a_2 \sqrt{1 - a_2^2}\, B_2 = a_2(a_1 A_3 - a_3 A_1) \tag{3-b}$$

$$a_3 \sqrt{1 - a_3^2}\, B_3 = a_3(a_2 A_1 - a_1 A_2) \ . \tag{3-c}$$

Adding these expressions yields one result:

$$a_1 \sqrt{1 - a_1^2}\, B_1 + a_2 \sqrt{1 - a_2^2}\, B_2 + a_3 \sqrt{1 - a_3^2}\, B_3 = 0 \tag{4}$$

Or they may be solved in pairs in order to obtain expressions for A_1, A_2, and A_3 in terms of A, B_1, B_2, and B_3. For example, if (3-b) and (3-c) are solved for A_2 and A_3, and substituted into (1), then

$$A = a_1A_1 + a_2 \frac{-\sqrt{1 - a_3^2}\, B_3 + a_2A_1}{a_1} + a_3 \frac{\sqrt{1 - a_2^2}\, B_2 + a_3A_1}{a_1} .$$

Multiplication by a_1 and collection of terms then leads to

$$-(a_1^2 + a_2^2 + a_3^2)A_1 = -a_1A + a_3\sqrt{1 - a_2^2}\, B_2 - a_2\sqrt{1 - a_3^2}\, B_3$$

or

$$A_1 = a_1A - a_3\sqrt{1 - a_2^2}\, B_2 + a_2\sqrt{1 - a_3^2}\, B_3, \tag{5-a}$$

using the condition on (1). In a similar manner relations for A_2 and A_3 are obtained:

$$A_2 = \frac{1}{a_1}\left\{a_1a_2A - a_2a_3\sqrt{1 - a_2^2}\, B_2 - (1 - a_2^2)\sqrt{1 - a_3^2}\, B_3\right\} \tag{5-b}$$

$$A_3 = \frac{1}{a_1}\left\{a_1a_3A + (1 - a_3^2)\sqrt{1 - a_2^2}\, B_2 + a_2a_3\sqrt{1 - a_3^2}\, B_3\right\} . \tag{5-c}$$

Now if the point A_0 is permitted to move along the direction A_0A, then dA_0 will be a linear combination of A_0 and A:

$$dA_0 = \lambda A_0 + \mu A .$$

However,

$$(A_0dA_0) = \lambda(A_0A_0) + \mu(A_0A),$$

which gives $\lambda = 0$ since $(A_0A_0) = 1$ (therefore, $(A_0dA_0) = 0$) and A_0 is on the plane $(AX) = 0$. Using this fact

$$(AdA_0) = \mu(AA) = \mu$$

so that dA_0 can be written as

$$dA_0 = (AdA_0)\ A.$$

Taking the scalar product of both sides of this equation with B_2 and B_3 gives

$$(dA_0B_2) = (dA_0B_3) = 0$$

because, for instance,

$$(dA_0B_2) = (AdA_0)(AB_2) = 0$$

since B_2 is on the plane $(AX) = 0$. These facts, along with the equation (5-a) derived above for A_1, yield

$$(dA_0A_1) = a_1(dA_0A) - a_3\sqrt{1 - a_2^2}\ (dA_0B_2) + a_2\sqrt{1 - a_3^2}\ (dA_0B_3)$$

$$(dA_0A_1) = a_1(dA_0A). \tag{6}$$

If ω represents the angle between A_0A and the normal A_0A_1 to E, then

$$\cos\omega = (AA_1) = a_1(A_1A_1) + a_2(A_2A_1) + a_3(A_3A_1),$$

or

$$\cos\omega = a_1. \tag{7}$$

Furthermore, if dF' represents the density of the point A_1 on the plane $A_1A_2A_3$, by Theorem II-1

$$dF' = (dA_1A_2)(dA_1A_3) = (dA_2A_1)(dA_3A_1); \tag{8}$$

dF' can be thought of geometrically as denoting the area element described by the point A_1 moving about on the plane $A_1A_2A_3$ whose pole is A_0.

Having provided all this information, the proof of the following theorem is easy.

Theorem III-5. If $ds = (dA_0A)$, so that ds is an element of length on A_0A, then $dE = \cos\omega \, ds \, dF'$.

Proof. This follows because $E = A_0A_1A_2$ and

$$dE = (dA_0A_1)(dA_2A_1)(dA_3A_1);$$

hence

$$dE = a_1(dA_0A)(dA_2A_1)(dA_3A_1)$$

by (6), and

$$dE = \cos\omega \, ds \, dF'$$

by (7) and (8). Q. E. D.

As an application of this theorem, the number of planes which cut a given curve C can be calculated. First note that identification of the endpoints of the diameters of a great circle of a unit sphere provides a model for the elliptic plane. Then temporarily fixing the point A

$$\int_{\substack{\text{elliptic}\\ \text{plane}}} \cos\omega \, dF' = \int_{\substack{\text{hemi-}\\ \text{sphere}}} \cos\omega \, dF' = \int_{\substack{\text{disc}\\ S_1}} \sec\omega \cos\omega \, dS_1$$

$$= \int_{\text{disc } S_1} dS_1 = \pi$$

using the theory of surface integrals and denoting by dS_1 the area element on the disc S_1. If L is the length of the curve C, then

$$\int n\, dE = \int \cos\omega\, ds\, dF' = \int ds \int \cos\omega\, dF' = \pi L,$$

where n is the number of intersections of the plane E and the curve C.

This result can be specialized. If C is a line, then $n = 1$ and $L = \pi$, giving

$$\int dE = \pi \cdot \pi = \pi^2.$$

This represents the "number" of planes in elliptic 3-space. From the duality of a point and its polar, it also represents the number of points in elliptic 3-space.

As mentioned previously there is a result in 2-space analogous to Theorem III-5. Let K be a curve with a tangent at every point. Suppose that a tangent T contacts K at x and a straight line G intersects K at x also. Then if ϕ is the elliptic angle between T and K and ds the distance between x and $x + dx$,

$$dG = \left|\sin\phi\right| ds\, d\phi.$$

Furthermore, this analogy can be specialized in a manner similar to that used above.

(a) If G intersects K in n points, then

$$\int n\, dG = 2L$$

where L is the Cayley length of K.

(b) If K is a straight line, then n = 1 so that

$$\int dG = 2\pi$$

and the measure of all straight lines in the Cayley plane is 2π.

(c) Duality in the plane and (b) give the measure of points as 2π also; i.e., the area of the elliptic plane is 2π.

Now attention is shifted to a situation similar to that just discussed, the case where lines intersect a surface. Sought is an expression for dG which is similar to that found for dE in Theorem III-5.

Again a great many preliminaries are necessary. A point A is chosen as before (on the plane $A_1A_2A_3$) and A_0 is permitted to move. The previously developed expressions for A_2 and A_3 (5-b) and (5-c) hold and they yield

$$(dA_0A_2) = \frac{1}{a_1}\left\{a_1a_2(dA_0A) - a_2a_3\sqrt{1 - a^2}\,(dA_0B_2)\right.$$
$$\left. - (1 - a_2^2)\sqrt{1 - a_3^2}\,(dA_0B_3)\right\},$$
$$(dA_0A_3) = \frac{1}{a_1}\left\{a_1a_3(dA_0A) + (1 - a_3^2)\sqrt{1 - a_2^2}\right.$$
$$\left. + a_2a_3\sqrt{1 - a_3^2}\,(dA_0B_3)\right\}.$$

These relations in turn give

$$(dA_0A_2)(dA_0A_3)$$
$$= \frac{1}{a_1^2}\left\{(1-a_2^2)^{3/2}(1-a_3^2)^{3/2}(dA_0B_2)(dA_0B_3)\right.$$
$$- a_2^2a_3^2\sqrt{1-a_2^2}\sqrt{1-a_3^2}\,(dA_0B_2)(dA_0B_3)$$

$$+ a_1a_2^2a_3\sqrt{1-a_3^2}\ (dA_0A)(dA_0B_3)$$

$$+ a_1a_2(1-a_3^2)\sqrt{1-a_2^2}\ (dA_0A)(dA_0B_2)$$

$$+ a_1a_2a_3^2\sqrt{1-a_2^2}\ (dA_0A)(dA_0B_2)$$

$$+ a_1a_3(1-a_2^2)\sqrt{1-a_3^2}\ (dA_0A)(dA_0B_3)\Big\}$$

$$= \frac{1}{a_1^2}\ \Big\{\Big((1-a_2^2)(1-a_3^2) - a_2^2a_3^2\sqrt{1-a_2^2}\sqrt{1-a_3^2}\ (dA_0B_2)(dA_0B_3)$$

$$+ (dA_0A)\Big[\Big(a_1a_2^2a_3\sqrt{1-a_3^2} + a_1a_3(1-a_2^2)\sqrt{1-a_3^2}\Big)\ (dA_0B_3)$$

$$+ \Big(a_1a_2(1-a_3^2)\sqrt{1-a_2^2} + a_1a_2a_3^2\sqrt{1-a_2^2}\Big)\ (dA_0B_2)\Big]\Big\}$$

$(dA_0A_2)(dA_0A_3)$

$$= \frac{1}{a_1^2}\Big\{(1-a_2^2-a_3^2)\sqrt{1-a_2^2}\sqrt{1-a_3^2}\ (dA_0B_2)(dA_0B_3)$$

$$+ (dA_0A)\Big[a_1a_3\sqrt{1-a_3^2}\ (dA_0B_3) + a_1a_2\sqrt{1-a_2^2}\ (dA_0B_2)\Big]\Big\}\ .$$

However, by (4)

$$a_1\sqrt{1-a_1^2}\ (dA_0B_1) + a_2\sqrt{1-a_2^2}\ (dA_0B_2)$$

$$+ a_3\sqrt{1-a_3^2}\ (dA_0B_3) = 0$$

so that

$$-a_1^2\sqrt{1-a_1^2}\ (dA_0B_1) = a_1a_2\sqrt{1-a_2^2}\ (dA_0B_2)$$

$$+ a_1a_3\sqrt{1-a_3^2}\ (dA_0B_3).$$

Substituting this in the above gives

$$(dA_0A_2)(dA_0A_3) = \frac{1}{a_1^2}\left\{a_1^2\sqrt{1-a_2^2}\sqrt{1-a_3^2}\ (dA_0B_2)(dA_0B_3) + (dA_0A)(-a_1^2\sqrt{1-a_1^2})(dA_0B_1)\right\}$$

$$(dA_0A_2)(dA_0A_3) = \sqrt{1-a_2^2}\sqrt{1-3_3^2}\ (dA_0B_2)(dA_0B_3) - \sqrt{1-a_1^2}\ (dA_0A)(dA_0B_1). \qquad (9)$$

In view of what is being attempted, another interpretation for $(dA_0B_2)(dA_0B_3)$ and $(dA_0A)(dA_0B_1)$ is now required. The first of these looks a great deal like a point density in a plane, but it is not quite since A_0, B_2, and B_3 do not form a self-conjugate triangle. This can be easily checked by using (2):

$$B_2 = \frac{a_1}{\sqrt{1-a_2^2}} A_3 - \frac{a_3}{\sqrt{1-a_2^3}} A_1, \quad B_3 = \frac{a_2}{\sqrt{1-a_3^2}} A_1 - \frac{a_1}{\sqrt{1-a_3^2}} A_2.$$

Hence

$$(B_2B_3) = \frac{-a_2a_3}{\sqrt{1-a_2^2}\ \sqrt{1-a_3^2}} \neq 0 \qquad (10)$$

so that B_2 and B_3 are not conjugate points. Nevertheless, a self-conjugate triangle can be obtained by using A_0, either B_2 or B_3, and some point on the line joining B_2 and B_3. So suppose that A_0, B_3, and $\mu B_2 + \lambda B_3$ form a self-conjugate triangle. Then two of the conditions on these points are:

$$(B_3(\mu B_2 + \lambda B_3)) = 0$$

$$((\mu B_2 + \lambda B_3)(\mu B_2 + \lambda B_3)) = 1$$

The first condition gives, upon multiplication,

$$\mu(B_3B_2) + \lambda(B_3B_3) = \mu(B_2B_3) + \lambda = 0$$

or

$$\lambda = \frac{\mu a_2 a_3}{\sqrt{1 - a_2^2}\sqrt{1 - a_3^2}}$$

using (10). The second condition yields

$$\mu^2 + \lambda^2 + 2\lambda\mu(B_2B_3) = 1$$

so that

$$\mu^2 + \left(\frac{\mu a_2 a_3}{\sqrt{1-a_2^2}\sqrt{1-a_3^2}}\right)^2 + 2\mu\,\frac{-a_2a_3}{\sqrt{1-a_2^2}\sqrt{1-a_3^2}}$$

$$\cdot\,\frac{\mu a_2 a_3}{\sqrt{1-a\ ^2}\sqrt{1-a\ ^2}} = 1$$

or solving for μ,

$$\mu = \frac{\sqrt{1-a_2^2}\sqrt{1-a_3^2}}{a_1}\ .$$

Thus three points on the plane $A_1B_2B_3$ which form a self-conjugate triangle are A_0, B_3, and $\dfrac{\sqrt{1-a_2^2}\sqrt{1-a_3^2}}{a_1}\,B_2 + \dfrac{a_2a_3}{a_1}\,B_3$. Then the point density about A_0 by Theorem II-1 is dF, say, where

$$dF = \left(dA_0\ \left(\frac{\sqrt{1-a\ ^2}\sqrt{1-a\ ^2}}{a_1}\,B_2 + \frac{a_2a_3}{a_1}\,B_3\right)\right)\ (dA_0B_3)$$

$$dF = \frac{\sqrt{1-a_2^2}\sqrt{1-a_3^2}}{a_1}(dA_0B_2)(dA_0B_3) + \frac{a_2a_3}{a_1}(dA_0B_3)(dA_0B_3)$$

$$dF = \frac{\sqrt{1-a_2^2}\sqrt{1-a_3^2}}{a_1}(dA_0B_2)(dA_0B_3) \tag{11}$$

The second expression in (9), $(dA_0A)(dA_0B_1)$, is a point density as A_0, A, and B_1 form a self-conjugate triangle. Let

$$dS = (dA_0A)(dA_0B_1) \tag{12}$$

With these preparations there is another easily proved theorem.

Theorem III-6. If ω is the angle between $G = A_0A_1$ and the line A_0A, and dF' is as before (see Theorem III-5), then $dG = (\cos\omega\, dF - \sin\omega\, dS)\, dF'$.

Proof. From Theorem III-2

$$dG = (dA_0A_2)(dA_0A_3)(dA_1A_2)(dA_1A_3)$$

$$dG = \left[\sqrt{1-a_2^2}\sqrt{1-a_3^2}\,(dA_0B_2)(dA_0B_3) - \sqrt{1-a_1^2}\,(dA_0A)(dA_0B_1)\right](dA_1A_2)(dA_1A_3)$$

by (9);

$$dG = (a_1 dF - \sqrt{1-a_1^2}\, dS)(dA_1A_2)(dA_1A_3)$$

by (11) and (12);

$$dG = (\cos\omega\, dF - \sin\omega\, dS)\, dF'$$

by (7) and (8). Q. E. D.

Just as in the case of Theorem III-5, this formula can yield

significant information. As a first specialization, if A_0 is permitted to move only on $(AX) = 0$, the area described on the plane A_0AB_1 is zero; or, in other words, $dS = 0$, so that

$$dG = \cos \omega \, dF \, dF'.$$

Now if F is a surface with surface area Q, in the manner previously used

$$\int dG = \int \cos \omega \, dF \, dF' = \int dF \cos \omega \, dF' = Q\pi.$$

Thus

$$\int n \, dG = \pi Q$$

where n is the number of intersections of the line G with the surface F whose surface area is Q. One last consideration is that if F is a plane so that $n = 1$ and $Q = 2\pi$, then

$$\int dG = 2 \pi^2.$$

This calculation gives the "number" of lines in elliptic 3-space.

As mentioned in the Preliminaries, there are some formulas which appear both in Euclidean and elliptic spaces. The following calculations are a derivation of one such result involving the kinematic density. Recall that in Theorem III-4, the kinematic density was formed by the product of six terms. The three terms (dA_0A_i), $i = 1, 2, 3$ can be thought of as representing an infinitesimal displacement of A_0 in the respective direction A_0A_i, the directions being mutually orthogonal. Furthermore, the other terms represent infinitesimal rotations about these

three directions. The intermediate result desired here is a generalization: the displacements may be taken along three non-orthogonal directions, and so may the rotations. The final result sought is the so-called "basic formula".

Consider a fixed surface F_0, a moving surface F_1, and a point A_0 on their curve of intersection, C_{01}. Let A_0 move along this curve. Consider the two self-conjugate tetrahedrons $A_0A_1A_2A_3$ and $A_0D_1D_2A_3$ where A_0A_1 and A_0D_1 are the surface normals for F_0 and F_1, respectively. Represent the angle between A_0A_1 and A_0D_1 by $\sphericalangle$. Furthermore, (for later use) let

$$U_1 = A_3 \qquad V_1 = A_3$$
$$U_2 = A_2 \qquad V_2 = A_1$$
$$U_3 = D_2 = \mu A_1 + \lambda A_2 \qquad V_3 = D_1 = \beta A_1 + \delta A_2$$

The μ, λ, β, and δ can be calculated from the above using the fact that the distance between two conjugate points on the elliptic plane is $\pi/2$; for example,

$$D_1 = \beta A_1 + \delta A_2$$

so that

$$(A_1D_1) = \beta(A_1A_1) + \delta(A_1A_2) = \beta.$$

However,

$$(A_1D_1) = \cos \sphericalangle$$

because $\sphericalangle$ is the distance between A_1 and D_1. Moreover,

$$(A_2D_1) = \beta(A_2A_1) + \delta(A_2A_2) = \delta$$

and

$$(A_2D_1) = \cos(\pi/2 - \alpha) = \sin\alpha .$$

Thus

$$D_1 = \cos\alpha\, A_1 + \sin\alpha\, A_2,$$

and in a similar manner

$$D_2 = -\sin\alpha\, A_1 + \cos\alpha\, A_2.$$

Now the alternate expressions for the three (dA_0A_i) terms of the kinematic density will be calculated. Considering (6) if $A = a_1A_1 + a_2A_2 + a_3A_3$, then the infinitesimal displacement $ds = (dA_0A)$ of A_0 along A_0A satisfies

$$(dA_0A_i) = a_i(dA_0A) = a_i ds.$$

Thus the infinitesimal displacements along A_0U_1, A_0U_2, A_0U_3 (for $A = U_1 = A_3$, $A = U_2 = A_2$, $A = U_3 = -\sin\alpha\, A_1 + \cos\alpha\, A_2$) are:

$$ds_1 = (dA_0U_1) = (dA_0A_3). \tag{13-a}$$

$$ds_2 = (dA_0U_2) = (dA_0A_2), \tag{13-b}$$

and

$$-\sin\alpha\, ds_3 = -\sin\alpha(dA_0U_3) = (dA_0A_1). \tag{13-c}$$

Therefore

$$(dA_0A_3)(dA_0A_2)(dA_0A_1) = -\sin\alpha\, ds_1ds_2ds_3$$

or

$$(dA_0A_1)(dA_0A_2)(dA_0A_3) = \sin \alpha \; ds_1 ds_2 ds_3. \tag{14}$$

Note that while the directions A_0A_1, A_0A_2, A_0A_3 are mutually orthogonal, the directions A_0U_1, A_0U_2, A_0U_3 are not.

Consider once again the plane $(AX) = 0$ though the point A_0, and the infinitesimal rotation $d\alpha$ about the direction A_0A which is defined by

$$d\alpha = (dB_2B) \tag{15}$$

where B is selected to be a point on the line B_2B_3 which is orthogonal to B_2. Again if $B = \mu B_2 + \lambda B_3$ as preceding Theorem III-6, then

$$(BB) = \mu^2 + \lambda^2 + 2\mu\lambda(B_2B_3) = 1$$

and

$$(BB_2) = \mu(B_2B_2) + \lambda(B_2B_3) = \mu + \lambda(B_2B_3) = 0.$$

Now by (10)

$$(B_2B_3) = \frac{-a_2a_3}{\sqrt{1 - a_2^2}\sqrt{1 - a_3^2}}$$

so that

$$\left(\frac{\lambda a_2a_3}{\sqrt{1-a_2^2}\sqrt{1-a_3^2}}\right)^2 + \lambda^2 - 2\left(\frac{+a_2a_3\lambda}{\sqrt{1-a_2^2}\sqrt{1-a_3^2}}\right)^2 = 1$$

$$(a_2^2a_3^2 + (1-a_2^2)(1-a_3^2) - 2a_2^2a_3^2)\lambda^2 = (1-a_2^2)(1-a_3^2),$$

or

$$\lambda = \frac{\sqrt{1-a_2^2}\sqrt{1-a_3^2}}{a_1}$$

by the condition on (1). Therefore

$$(dB_2B_3) = \left(dB_2(\frac{1}{\lambda} B - \frac{\mu}{\lambda} B_2)\right)$$

$$= \frac{1}{\lambda} (dB_2B) - \frac{\mu}{\lambda} (dB_2B_2)$$

$$= \frac{1}{\lambda} (dB_2B) = \frac{a_1}{\sqrt{1-a_2^2}\sqrt{1-a_3^2}}(dB_2B) = \frac{a_1}{\sqrt{1-a_2^2}\sqrt{1-a_3^2}} d\alpha.$$

Now A is fixed by a rotation about the direction A_0A so that from (5-a)

$$dA_1 = 0 - a_3\sqrt{1-a_2^2}\, dB_2 + a_2\sqrt{1-a_3^2}\, dB_3 .$$

Using (5-b), this yields

$$(dA_1A_2) = (-a_3\sqrt{1-a_2^2}\, dB_2 + a_2\sqrt{1-a_3^2}\, dB_3)$$

$$\frac{1}{a_1}\left(a_1a_2A - a_2a_3\sqrt{1-a_2^2}\, B_2 - (1-a_2^2)\sqrt{1-a_3^2}\, B_3\right)$$

$$(dA_1A_2) = -a_2a_3\sqrt{1-a_2^2}\,(dB_2A) + \frac{a_3}{a_1}(1-a_2^2)\sqrt{1-a_2^2}\sqrt{1-a_3^2}\,(dB_2B_3)$$

$$+ a_2^2\sqrt{1-a_3^2}\,(dB_3A) - \frac{a_2^2a_3}{a_1}\sqrt{1-a_2^2}\sqrt{1-a_3^2}\,(dB_3B_2).$$

Two additional statements are now required. First, because B_2 is on the plane $(AX) = 0$, $(AB_2) = 0$. Then in an infinitesimal rotation about A_0A, A is fixed so that

$$d(AB_2) = 0 = (dB_2A) + (B_2dA) = (dB_2A).$$

Thus $(dB_2A) = 0$; in a similar manner $(dB_3A) = 0$. Moreover, from (10)

$$(B_2B_3) = \frac{-a_2a_3}{\sqrt{1-a_2^2}\sqrt{1-a_3^2}}$$

so that

$$(dB_2B_3) + (B_2dB_3) = 0.$$

Now the expression for (dA_1A_2) may be rewritten as

$$\begin{aligned}(dA_1A_2) &= \frac{a_3-a_2^2a_3}{a_1}\sqrt{1-a_2^2}\sqrt{1-a_3^3}\ (dB_2B_3)\\ &\qquad + \frac{a_2^2a_3}{a_1}\sqrt{1-a_2^2}\sqrt{1-a_3^2}\ (dB_2B_3)\\ &= \frac{a_3}{a_1}\sqrt{1-a_2^2}\sqrt{1-a_3^2}\ (dB_2B_3)\\ &= \frac{a_3}{a_1}\sqrt{1-a\ ^2}\sqrt{1-a\ ^2}\ \frac{a_1}{\sqrt{1-a_2^2}\sqrt{1-a_3^2}}\,d\alpha\ ,\end{aligned}$$

or

$$(dA_1A_2) = a_3\ d\alpha\ .$$

Two other formulas may be obtained through a cyclic permutation of 1, 2, 3:

$$(dA_iA_j) = a_kd\alpha \qquad (i,\ j,\ k) = (1,\ 2,\ 3). \tag{16}$$

With this relation (16) the infinitesimal rotations about the lines A_0V_1, A_0V_2, and A_0V_3 can be calculated; call these $d\alpha_1$, $d\alpha_2$, and $d\alpha_3$, respectively. For calculating $d\alpha_1$, consider

$$A = a_1A_1 + a_2A_2 + a_3A_3 = V_1 = A_3.$$

Thus

$$d\alpha_1 = (dA_1A_2). \qquad (17\text{-a})$$

In a similar manner, it follows that ($A = V_2 = A_1$).

$$d\alpha_2 = (dA_2A_3); \qquad (17\text{-b})$$

likewise, since

$$V_3 = D_1 = \cos\alpha\, A_1 + \sin\alpha\, A_2$$

$$(dA_3A_1) = a_2 d\alpha_3 = \sin\alpha\, d\alpha_3\ . \qquad (17\text{-c})$$

Thus multiplying these results together yields,

$$(dA_1A_2)(dA_2A_3)(dA_3A_1) = d\alpha_1 d\alpha_2\ (\sin\alpha\, d\alpha_3)$$

$$(dA_1A_2)(dA_2A_3)(dA_3A_1) = \sin\alpha\, d\alpha_1 d\alpha_2 d\alpha_3. \qquad (18)$$

Now the kinematic density for the surface F can be easily calculated.

Theorem III-7. The kinematic density for the surface F_0 can be written

$$dF_0 = \sin^2\alpha\ ds_1\ ds_2\ ds_3\ d\alpha_1\ d\alpha_2\ d\alpha_3.$$

Proof. By Theorem III-4,

$$dF_0 = (dA_0A_1)(dA_0A_2)(dA_0A_3)(dA_1A_2)(dA_1A_3)(dA_2A_3);$$

hence

$$dF_0 = \sin\alpha\ ds_1\ ds_2\ ds_3\ \sin\alpha\ d\alpha_1\ d\alpha_2\ d\alpha_3$$

by (14) and (18).

$dF_0 = \sin^2 \alpha\, ds_1\, ds_2\, ds_3\, d\alpha_1\, d\alpha_2\, d\alpha_3$. Q. E. D.

A similar expression can be obtained for the surface F_1.

In order to obtain the so-called basic formula, the definition of a line element is needed. Consider a surface F and a point A_0 which is moving on F. If $\overline{A_0A_3}$ is a directed tangent to F at A_0 with $(A_0A_3) = 0$, then $\overline{A_0A_3}$ is a line element L through A_0 on F. Furthermore, if $A_0A_1A_2A_3$ is a self-conjugate tetrahedron so that A_0A_1 is a surface normal to F at A_0, and if dt_2, dt_3 represent the displacements of the point A_0 in the directions of A_2 and A_3, and if $d\tau_1$ represents the infinitesimal rotation about A_0A_1, then

$$dL = dt_2\, dt_3\, d\tau_1 \tag{19-a}$$

is defined to be the density of the line elements L on F.

Continuing the previous discussion, denote the density of the line elements on F_0 and F_1 by

$$dL_0 = dt_2\, dt_3\, d\tau_1, \qquad dL_1 = dt_2^*dt_3^*d\tau_1^*$$

respectively. Now A_0 was chosen to be a point moving on C_{01}, the curve of intersection of F_0 and F_1. Furthermore, assume now that $L = \overline{A_0A_3}$ is a common line element for both surfaces. $A_0A_1A_2A_3$ is the self-conjugate tetrahedron and A_0A_1 is the normal for F_0; $A_0D_1D_2A_3$ is the self-conjugate tetrahedron and A_0D_1 is the normal for F_1. Now it is evident from the previous definitions and the statements cited that

$$dt_2 = ds_2 \quad (13\text{-b}), \qquad dt_2^* = ds_3 \quad (13\text{-c}),$$
$$dt_3 = ds_1 \quad (13\text{-a}), \qquad dt_3^* = ds_1 \quad (13\text{-a}),$$
$$d\tau_1 = d\alpha_2 \quad (17\text{-b}), \qquad d\tau_1^* = d\alpha_3 \quad (16) \text{ and } (17\text{-c}).$$

If ds_1 is denoted by ds, the following theorem, the basic formula, follows from the above.

Theorem III-8. $dF_1\, ds = \sin^2 \alpha\, dL_0\, dL_1\, d\alpha_1$.

Proof. From Theorem III-7,

$$\begin{aligned} dF_1\, ds &= \sin^2 \alpha\, ds_1\, ds_2\, ds_3\, d\alpha_1\, d\alpha_2\, d\alpha_3\, ds \\ &= \sin^2 \alpha\, dt_3^*\, dt_2\, dt_2^*\, d\alpha_1\, d\tau_1\, d\tau_1^*\, dt_3 \\ &= -\sin^2 \alpha\, dt_2\, dt_3\, d\tau_1\, dt_2^*\, dt_3^*\, dt_1^*\, d\alpha_1 \\ &= -\sin^2 \alpha\, dL_0\, dL_1\, d\alpha_1 \end{aligned}$$

by (19-b). Because only positive densities are considered, the desired result follows. Q. E. D.

The next matter of importance is to prove the principal kinematic formula. In that proof, however, it will be necessary to use some integral formulas which have not been mentioned previously. For this reason, some of these will be developed first; for the sake of completeness, some other formulas will be included which involve other invariants.

For three basic integral formulas recall that in Theorem III-4 the kinematic density was found to be

$$dK = (dA_0A_1)(dA_0A_2)(dA_0A_3)(dA_2A_3)(dA_3A_1)(dA_1A_2),$$

where $A_0A_1A_2A_3$ is a self-conjugate tetrahedron. Note the form of the first three terms (see Theorem III-1):

$$(dA_0A_1)(dA_0A_2)(dA_0A_3) = dA.$$

Therefore, define

$$dK_A = (dA_2A_3)(dA_3A_1)(dA_1A_2)$$

so that

$$dK = dA\ dK_A.$$

Similarly, upon reviewing Theorems III-2 and III-3, define in the obvious way dK_G and dK_E and write

$$dK = dG\ dK_G$$
$$dK = dE\ dK_E$$

without regard to sign. (Disregard signs in this discussion as all densities are considered to be positive.)

Next, consider

$$\int dK_A = \int (dA_2A_3)(dA_3A_1)(dA_1A_2)$$
$$= \int (dA_1A_3)(dA_1A_2) \int (dA_2A_3).$$

If dK_A is integrated in this form over all possible positions in the space, note that $(dA_1A_3)(dA_1A_2)$ is the area element about the point A_1. Furthermore, (dA_2A_3) is an infinitesimal rotation about the point A_1 and

$$\int dK_A = 2\pi \int (dA_1A_3)(dA_1A_2);$$

in a rotation of 2π about A_0A_1, the plane is covered twice so that

$$\int dK_A = 2\pi \cdot 2\pi \cdot 2 = 8\,\pi^2. \tag{20-a}$$

Another formula is easily obtained, note that

$$\int dK_G = \int (dA_0A_1)(dA_2A_3).$$

These integrals involve the infinitesimal rotation (dA_2A_3) about A_1 and the infinitesimal displacement (dA_1A_0) of A_1 along the line A_1A_0; hence

$$\int dK_G = \pi \cdot 2\pi \cdot 2 = 4\,\pi^2. \tag{20-b}$$

(The line A_0A_1 is traced twice.) The third formula is easier yet:

$$\int dK_E = 8\,\pi^2 \tag{20-c}$$

because it is the dual of the first integral considered.

Two formulas involving volume and surface area may be derived from these relations. Let K_0 and K_1 be two regions with K_0 fixed and K_1 moving. Also let V_i and $Q_i (i = 0, 1)$ represent the volume and surface area for K_i; the volume and surface area of $K_{01} = K_0 \cap K_1$ are V_{01} and Q_{01}. The kinematic density on K_1 is dK_1. Then

$$\int V_{01}\, dK_1 = \int\left(\int_{A\in K_0\cap K_1} dA\right) dK_1,$$

fixing a position of K_1

$$= \int_{A\in K_0\cap K_1} dA\, dK_1$$

$$= \int_{A\in K_0}\left(\int_{A\in K_1} dK_1\right) dA,$$

where A is first regarded as fixed on K_0,

$$\int V_{01}\, dK_1 = 8\,\pi^2\, V_1 \int_{A\in K_0} dA$$

by (20-2), or

$$\int V_{01}\, dK_1 = 8\,\pi^2\, V_0 V_1. \qquad (21)$$

In a similar manner another formula can be obtained. The equation

$$\int Q_{01}\, dK_1 = \int Q(R_1\cap K_0)\, dK_1 + \int Q(R_0\cap K_1)\, dK_1$$

can be simplified further; for example, if dQ is an area element,

$$\int Q(R_1\cap K_0)\, dK_1 = \int\left(\int_{A\in R_1\cap K_0} dQ\right)\, dK_1$$

$$= \int_{A\in R_1\cap K_0} dQ\, dK_1$$

$$= \int_{A\in R_1}\left(\int_{A\in K_0} dK\right)\, dQ,$$

fixing A on R_1,

$$= 8\,\pi^2\, V_0 \int_{A\in R_1} dQ = 8\,\pi^2\, V_0 Q_1.$$

Likewise,

$$\int Q(F_0 \cap K_1)\,dK_1 = 8\,\pi^2\,V_1Q_0$$

so that

$$\int Q_{01}dK_1 = 8\,\pi^2\,(V_0Q_1 + Q_0V_1). \qquad (22)$$

To complete this listing of integral formulas, integrals involving arc length of a curve and mean curvature of a surface will be considered. The first formula to be obtained is a direct application of Theorem III-8.

For this first result, let F_0 and F_1 be two smooth surfaces with surface areas Q_0 and Q_1, and curve of intersection C_{01}, of length L_{01}. Considering F_0 as fixed and F_1 as moving with kinematic density dF_1 gives (from Theorem III-8):

$$\int L_{01}dF_1 = \int \sin^2 \alpha \; dL_0 \; dL_1 \; d\alpha.$$

However, according to (19), dL_i can be broken up. For example,

$$dL_1 = dt_2 \; dt_3 \; d\tau_1 = dQ_0 \; d\tau_1$$

where dQ_0 is an area element about the point on the surface. Thus

$$\int L_{01}dF_1 = \int_0^{\pi} \sin^2 \alpha \; d\alpha \int dQ_0 \int_0^{2\pi} d\tau_1 \int dQ_1 \int_0^{2\pi} d\tau_1{}^{*}$$

$$\int L_{01}dF_1 = \pi/2 \; Q_0 \; 2\pi \; Q_1 \; 2\pi = 2\,\pi^3\,Q_0Q_1. \qquad (23)$$

In preparation for the proof of the integral formula involving mean curvature, a definition and some formulas from differential geometry are needed. The integral

$$M_i = \int H_i \, dQ_i \qquad (24)$$

is defined to be the surface integral of the mean curvature H_i of a surface R_i. The quantity dQ_i represents, as before, the surface area element on R_i. Formulas that are required are

$$"\left|" A_0, A_1, dA_0, dA_1 "\right|" = -2H \, dQ \qquad (25\text{-a})$$

and

$$"\left|" A_0, A_1, dA_1, dA_1 "\right|" = 2K_r \, dQ,$$

both being analogies of similar results for Euclidean 3-space. Here $A_0A_1A_2A_3$ is again a self-conjugate tetrahedon but with A_0 and A_1 playing special roles: A_0 is a point on a surface and A_1 is the pole of the tangent plane to the surface at A_0. Furthermore, H is the mean curvature and K_r the relative curvature on the surface being considered. (The relative curvature will be dealt with more extensively later.) The element of surface area is represented by dQ. Moreover, the symbol $"|"$, , , $"|"$ indicates that in evaluating the determinant $|.\ ,\ ,\ |$, exterior products are used. Examplea of this will be seen in later calculations.

Now consider the following situation. Two regions K_0 (fixed) and K_1 (moving) are given which have smooth boundaries R_i with surface areas Q_i and mean curvatures H_i, $(i = 0, 1)$. Let V_i be the volume of K_i. Then if M_{01} is the surface integral of mean curvature on the boundary of $K_0 \cap K_1$, a formula for

$$\int M_{01} \, dK_1$$

as desired. Notice that this may be split up into three integrals: the quantity M_{01} is the sum of like quantities on $R_0 \cap K_1$, $K_0 \cap R_1$, and along the curve of intersection of R_0 and R_1. Thus

$$\int M_{01} dK_1 = \int M(R_0 \cap K_1) dK_1 + \int M(K_0 \cap R_1) dK_1 + J_3 \qquad (26)$$

where J_3 is equal to the integrated effect B of all the indeterminant normals $A_0 A_1$ (see below) along the curve of intersection, C_{01}.

The calculation of the first two integrals is by far the easiest part. Consider, for example,

$$\int M(K_0 \cap R_1) dK_1 = \int \left(\int_{A \in K_0 \cap R_1} H_1 dQ_1 \right) dK_1$$

by (24)

$$= \int_{A \in K_0 \cap R_1} H_1 \, dQ_1 \, dK_1$$

$$= \int_{A \in R_1} \left(\int_{A \in K_0} dK_1 \right) H_1 \, dQ_1$$

where A is first fixed on R_1,

$$= 8 \pi^2 V_0 \int_{A \in R_1} H_1 \, dQ_1$$

or

$$\int M(K_0 \cap R_1) dK_1 = 8 \pi^2 V_0 M_1. \qquad (27)$$

By inversion (I-5),

$$\int M(R_0 \cap K_1) dK_1 = 8 \pi^2 V_1 M_0.$$

J_3 remains to be calculated. Let A_0 be a point on the curve of intersection C_{01}. A_3 is chosen to be conjugate to A_0 and on the tangent to C_{01} at A_0. Furthermore, for each tangent plane to the surface R_i at A_0, let $r_i (i = 0, 1)$ be the pole of that tangent plane. Then $r_0 r_1$ will be the polar line to $A_0 A_3$. Let 2θ $(0 \leq \theta \leq \pi/2)$ represent the distance between r_0 and r_1. (Hence, by (I-6), 2θ is also the angle between the tangent planes to R_1 and R_2 at A_0.) The segments $r_0 r_1$ each have a mid-point; let v and w be these. Then set

$$r_0 = \lambda v + \mu w$$

and calculate λ and μ by taking $(r_0 v)$ and $(r_0 w)$:

$$(r_0 v) = \lambda = \cos \theta$$

$$(r_0 w) = \mu = \cos(\theta + \pi/2) = -\sin \theta.$$

In a similar manner, an expression for r_1 can be obtained so that

$$r_0 = v \cos \theta - w \sin \theta$$

and (29)

$$r_1 = v \cos \theta + w \sin \theta.$$

Furthermore, points A_1 and A_2 can be chosen so that

$$A_1 = v \cos \phi + w \cos(\phi - \pi/2) = v \cos \phi + w \sin \phi$$
(30)
$$A_2 = v \cos(\phi + \pi/2) + w \cos \phi = -v \sin \phi + w \cos \phi$$

where $-\theta \leq \phi \leq \theta$. The points $A_0 A_1 A_2 A_3$ form a self-conjugate tetrahedron as can be readily verified; also A_1 is between r_0 and r_1.

Now because B takes into consideration the mean curvatures where the normals are indeterminant (between r_0 and r_1) B can be written

$$B = \int H\, dQ = -\frac{1}{2}\int "\Big|" A_0,\ A_1,\ dA_0,\ dA_1 "\Big|",$$

by (25-a). To calculate the expression on the right-hand side, dA_0 and dA_1 must be calculated. First, let

$$dA_0 = \lambda A_0 + \mu A_3.$$

Taking the scalar product of both sides with first A_0 and then A_3 shows that $\lambda = 0$ and $\mu = (dA_0A_3)$ so that

$$dA_0 = (dA_0A_3)\ A_3.$$

However, (dA_0A_3) represents an infinitesimal displacement along A_0A_3; calling this ds, then

$$dA_0 = ds\ A_3.$$

Second, from (30),

$$dA_1 = dv \cos \phi + dw \sin \phi + (-v \sin \phi + w \cos \phi)d\phi$$

$$= dv \cos \phi + dw \sin \phi + A_2\ d\phi.$$

With these calculations B may be rewritten

$$B = -\frac{1}{2}\int "\Big|" A_0,\ A_1,\ ds\ A_3,\ dv \cos \phi + dw \sin \phi + A_2\ d\phi "\Big|".$$

The points v and w, however, depend only on the parameter of the curve C_{01}, which is s. This means that dv and dw are multiples of ds; hence

$$B = -\frac{1}{2}\int "\left|" A_0, A_1, ds\, A_3, A_2\, d\phi "\right|"$$

$$= -\frac{1}{2}\int \left| A_0, A_1, A_3, A_2 \right| ds\, d\phi$$

$$= \frac{1}{2}\int ds\, d\phi$$

because $A_0A_1A_2A_3$ is a self-conjugate tetrahedron. Now J_3 can be calculated:

$$J_3 = \int B\, dK_1 = \frac{1}{2}\int ds\, dK_1\, d\phi$$

$$= \frac{1}{2}\int_{-\theta}^{\theta}\left(\int ds\, dK_1\right) d\phi$$

$$= \int \theta\, ds\, dK_1.$$

In Theorem III-8, let $2\theta = \alpha$ so that

$$J_3 = \frac{1}{2}\int \alpha \sin^2 \alpha\, dL_0\, dL_1\, d\alpha.$$

Calculating

$$\frac{1}{2}\int_0^{\pi} \alpha \sin^2 \alpha\, d\alpha = \frac{\pi^2}{8}$$

enables J_3 to be written

$$J_3 = \frac{\pi^2}{8}\int dL_0\, dL_1$$

$$= \frac{\pi^2}{8}\, 2\pi Q_0\, 2\pi Q_1$$

in the manner of (23), or

$$J_3 = \frac{\pi^2}{2} Q_0 Q_1. \tag{31}$$

From (25), (27), (28), and (31) the desired result is obtained:

$$\int M_{01} dK_1 = 8 \pi^2 (M_0 V_1 + \frac{\pi^2}{16} Q_0 Q_1 + V_0 M_1). \tag{32}$$

The task now is to develop the principal kinematic formula for elliptic 3-space. Such a development was first done by W. Blaschke; hence this result is also called Blaschke's fundamental formula. The derivation below parallels parts of Blaschke's derivation for the Euclidean case. Some other parts of it are a great deal like the proof of (32).

Consider as above two regions K_0 and K_1 which have smooth boundary surfaces R_0 and R_1. For their region of intersection, $K_{01} = K_0 \cap K_1$, call the boundary surface R_{01}. Sought is an expression for the integral

$$\int C_r(R_{01}) \, dK_1$$

where C_r is the total relative curvature on the surface R_{01}. The total relative curvature on an arbitrary smooth surface is given by C_r, where

$$C_r = \int K_r \, dQ \; ; \tag{33}$$

K_r is the relative curvature and dQ is the surface area element of the surface.

Now the total relative curvature on the entire surface can be broken down into three parts, just as can the surface itself. Therefore, consider

$$\int C_r(R_{01})\ dK_1 = J_1' + J_2' + J_3' \qquad (34)$$

where J_1' is related to the total relative curvature on $R_0 \cap K_1$, J_2' is related to the total relative curvature on $K_0 \cap R_1$, and J_3' will be related to the integrated effect B' of the indeterminant normals A_0A_1 (to be introduced later) along the curve of intersection itself. Call this curve C_{01}. Thus

$$J_1' = \int C_r(R_0 \cap K_1)\ dK_1\ ,$$

$$J_2' = \int C_r(K_0 \cap R_1)\ dK_1\ ,$$

and

$$J_3' = \int_{C_{01}} B'\ dK_1 .$$

The integral J_2' is easily calculated by using a previous result:

$$J_2' = \int C_r(K_0 \cap R_1)\ dK_1 = \int \Big(\int_{A \in K_0 \cap R_1} K_r\ dQ \Big)\ dK_1,$$

Thus

$$J_2' = \int_{A \in R_1} \Big(\int_{A \in K_0} dK \Big)\ K_r dQ,$$

if A is first fixed on R_1

$$= 8\ \pi^2\ V_0 \int_{A \in R_1} K_r\ dQ$$

by (20-a), or

$$J_2' = 8\ \pi^2\ V_0\ C_{r1}, \qquad (35)$$

letting C_{ri} be the total relative curvature on R_i, $(i = 0, 1)$. Furthermore, that

$$J_1' = \int C_r(R_0 \cap K_1)\ dK_1 = 8\ \pi^2\ C_{r0}V_1 \qquad (36)$$

follows by inversion from (35).

At this point, a similar procedure to that used in calculating J_3 of (26) can be carried out to start the calculation of J_3' of (34). In this case

$$B' = \int K_r \, dQ = \frac{1}{2} \int \text{"}\big|\text{"}\ A_0,\ A_1,\ dA_1,\ dA_1\ \text{"}\big|\text{"}$$

by (25-b). Hence

$$\begin{aligned} B' &= \frac{1}{2} \int \text{"}\big|\text{"}\ A_0,\ A_1,\ dv \cos \emptyset + dw \sin \emptyset + A_2\, d\emptyset, \\ &\qquad dv \cos \emptyset + dw \sin \emptyset + A_2 d\emptyset\ \text{"}\big|\text{"} \\ &= \frac{1}{2} \int \text{"}\big|\text{"}\ A_0,\ A_1,\ dv \cos \emptyset + dw \sin \emptyset, \\ &\qquad dv \cos \emptyset + dw \sin \emptyset + A_2 d\emptyset\ \text{"}\big|\text{"} \\ &\qquad + \frac{1}{2} \int \text{"}\big|\text{"}\ A_0,\ A_1,\ A_2 d\emptyset,\ dv \cos \emptyset + dw \sin \emptyset + A_2 d\emptyset\ \text{"}\big|\text{"} \\ &= \frac{1}{2} \int \text{"}\big|\text{"}\ A_0,\ A_1,\ dv \cos \emptyset + dw \sin \emptyset,\ A_2 d\emptyset\ \text{"}\big|\text{"} \\ &\qquad + \frac{1}{2} \int \text{"}\big|\text{"}\ A_0,\ A_1,\ A_2 d\emptyset,\ dv \cos \emptyset + dw \sin \emptyset\ \text{"}\big|\text{"} \\ &= \int \text{"}\big|\text{"}\ A_0,\ A_1,\ dv \cos \emptyset + dw \sin \emptyset,\ A_2 d\emptyset\ \text{"}\big|\text{"}. \end{aligned}$$

For further simplification, consider the fact that A_0A_3vw form a self-conjugate tetrahedron. Because of this, dv can be written as a linear combination of A_0, A_3, v, and w and the constants evaluated by taking scalar products of dv with v, A_0, A_3, and w. If

$$dv = aA_0 + bA_3 + cv + dw,$$

Then $(v\bar{v}) = 0$ so that $c = 0$, and

$$(A_0\ dv) = a,\ (A_3\ dv) = b,\ (wdv) = d.$$

Hence

$$dv = (dv\ A_3)\ A_3 + (dvw)\ w + (dv\ A_0)\ A_0.$$

In a similar manner,

$$dw = (dw\ A_3)\ A_3 + (dwv)\ v + (dw\ A_0)\ A_0$$

so that

$$B' = \int {}''\big|''\ A_0,\ A_1,\ (dvA_3)\ A_3 \cos\phi + (dvw)\ w \cos\phi$$

$$+ (dvA_0)\ A_0 \cos\phi + (dwA_3)\ A_3 \sin\phi + (dwv)\ v \sin\phi$$

$$+ (dwA_0)\ A_0 \sin\phi,\ A_2\ d\phi\ {}''\big|''$$

$$= \int {}''\big|''\ A_0,\ A_1,\ dvA_3)\ A_3 \cos\phi + (dwA_3)\ A_3 \sin\phi$$

$$+ (dvw)\ w \cos\phi - (dvw)\ v \sin\phi,\ A_2\ d\phi\ {}''\big|''\ .$$

This last statement follows because $(vw) = 0$ implies that $(vdw) = -(dvw)$. Continuing the evaluation of B', and considering (30), yields

$$B' = \int {}''\big|''\ A_0.\ A_1,\ ((dvA\ )\cos\phi + (dwA_3)\sin\phi)\ A_3 + (dvw)\ A_2,$$
$$A_2\ d\phi\ {}''\big|''$$

$$= \int {}''\big|''\ A_0,\ A_1,\ ((dvA_3)\cos\phi + (dwA_3)\sin\phi)\ A_3,\ A_2\ d\phi\ {}''\big|''$$

$$= \int \big|A_0.\ A_1.\ A_3,\ A_2\big|\ ((dvA_3)\cos\phi + (dwA_3)\sin\phi)\ d\phi$$

$$= -\int ((dvA_3)\cos\phi + (dwA_3)\sin\phi)\ d\phi.$$

Now B' can be written

$$B' = - \int_{A_0 \in C_{01}} ((dvA_3) \cos \phi + (dwA_3) \sin \phi)\, d\phi$$

$$= - \int_{A_0 \in C_{01}}^{-\theta \leq \phi \leq \theta} \int_{-\theta}^{\theta} (dvA_3) \cos \phi \, d\phi - \int_{A_0 \in C_{01}} \int_{-\theta}^{\theta} (dwA_3) \sin \phi \, d\phi$$

$$= -2 \int_{A_0 \in C_{01}} (dvA_3) \sin \theta.$$

Further simplification can be done when (29) is solved for v:

$$v = \frac{r_0 + r_1}{2 \cos \theta} ;$$

then v is differentiated yielding

$$B' = - \int_{A_0 \in C_{01}} ((dr_0 \; A_3) + (dr_1 \; A_3)) \tan \theta. \qquad (37)$$

Now a relation is needed from the Frenet Formulas in elliptic space. Consider a space curve and a point x on that curve; select points t, n, and b on the tangent, principal normal, and binormal, respectively, which are conjugate to x.

Then the Frenet relations are

$$\frac{dt}{ds} = \frac{n}{K} - x$$

$$\frac{db}{ds} = \frac{n}{T}$$

$$\frac{dn}{ds} = - \frac{t}{K} - \frac{b}{T}$$

where s is the arc length of the curve (i.e., ds is the line element along the curve), 1/K is the curvature, and 1/T is the torsion. In the context above, $t = A_3$ and $x = A_0$, so that the first relation states

$$dA_3 = \frac{n}{K}\, ds - A_0\, ds.$$

Recall that r_i ($i = 0, 1$) were on a polar line to A_0A_3 so that $(r_i\, A_3) = 0$; hence, from the above equation

$$(dr_i A_3) = -\,(r_i dA_3) = -\,\frac{(r_i n)}{K}\, ds + (r_i A_0) ds$$

$$(dr_i A_3) = -\,\frac{(r_i n)}{K}\, ds.$$

Furthermore, if k_{n_i} represents the normal curvature of C_{01} at A_0 on the surface R_i, then another relation from differential geometry is Meusnier's:

$$\frac{(r_i n)}{K} = k_{n_i} \; . \tag{39}$$

With these preparations, (37), (38), and (39) give

$$B' = -\int_{A_0 \in C_{01}} ((dr_0\, A_3) + (dr_1\, A_3)) \tan\theta$$

$$= \int_{A_0 \in C_{01}} \left(\frac{(r_0 n)}{K} + \frac{(r_1 n)}{K} \right) ds \tan\theta$$

$$= \int_{A_0 \in C_{01}} (k_{n_0} + k_{n_1}) \tan\theta\, ds \; .$$

If in Theorem III-8, α is selected so that $\alpha = 2\theta$, then

$$\int B' dK_1 = \int (k_{n_0} + k_{n_1}) \tan\theta \, ds \, dK_1$$

$$= \int (k_{n_0} + k_{n_1}) \sin^2 2\theta \frac{\sin\theta}{\cos\theta} dL \; dL \; 2d\theta$$

$$= 8 \int (k_{n_0} + k_{n_1}) \sin^3 \cos\theta \, dL_0 \, dL_1 \, d\theta.$$

At this point Euler's equation from differential geometry,

$$k_{n_i} = k_i' \cos^2 \tau_i + k_i'' \sin^2 \tau_i$$

where k_i' and k_i'' are principal curvatures of the surface R_i and τ_i is the angle between C_{01} and a principal direction on R_i can be used to obtain

$$\int B' dK_1 = 8 \int (k_0' \cos^2 \tau_0 + k_0'' \sin^2 \tau_0 + k_1' \cos^2 \tau_1 + k_1'' \sin^2 \tau_1) \sin^3\theta \cos\theta \, dL_0 \, dL_1 \, d\theta;$$

$$\int B' dK_1 = 8 \int (k_0' \cos^2 \tau_0 + k_0'' \sin^2 \tau_0 + k_1' \cos^2 \tau_1 + k_1'' \sin^2 \tau_1) \sin^3\theta \cos\theta \, dQ_0 \, d\tau_0 \, dQ_1 \, d\tau_1 \, d\theta$$

because from (19), $dL_0 = dQ_0 \, d\tau_0$ and $dL_1 = dQ_1 \, d\tau_1$.

Now some auxiliary calculations are necessary. Since $0 \leq \theta < \pi/2$

$$\int_0^{\pi/2} \sin^3\theta \cos\theta \, d\theta = \frac{1}{4}.$$

The τ_i may range through a full rotation, so that

$$\int_{-\pi}^{\pi} \sin^2 \tau_i \, d\tau_i = \int_{-\pi}^{\pi} \cos^2 \tau_i \, d\tau_i = \pi.$$

Then

$$\int B \, dK_1 = 8 \int_0^{\pi/2} \sin^3 \theta \cos \theta \, d\theta \int \left\{ \int_{-\pi}^{\pi} k_0' \cos^2 \tau_0 \, d\tau_0 \, d\tau_1 \right.$$

$$+ \int_{-\pi}^{\pi} k_0'' \sin^2 \tau_0 \, d\tau_0 \, d\tau_1 + \int_{-\pi}^{\pi} k_1' \cos^2 \tau_1 \, d\tau_0 \, d\tau_1$$

$$\left. + \int_{-\pi}^{\pi} k_1'' \sin^2 \tau_1 \, d\tau_0 \, d\tau_1 \right\} dQ_0 \, dQ_1$$

$$= 4 \pi^2 \left\{ \int (k_0' + k_0'') \, dQ_0 \, dQ_1 + \int (k_1' + k_1'') \, dQ_0 \, dQ_1 \right\},$$

or

$$\int B \, dK_1 = 8 \pi^2 (M_0 Q_1 + Q_0 M_1) \tag{40}$$

by (2[illegible]). (As before, Q_i represents the surface area of R_i.)

Now the pieces can be assembled easily for Blaschke's fundamental formula in elliptic 3-space.

<u>Theorem III-9</u>.

$$\int C_r (R_{01}) \, dK_1 = 8 \pi^2 (C_{r0} V_1 + M_0 Q_1 + Q_0 M_1 + V_0 C_{r1}).$$

<u>Proof</u>. From (34), (35), (36), and (40),

$$\int C_r (R_{01}) \, dK_1 = 8 \pi^2 C_{r0} V_1 + 8 \pi^2 (M_0 Q_1 + Q_0 M_1) + 8 \pi^2 V_0 C_{r1}$$

$$\int C_r (R_{01}) \, dK_1 = 8 \pi^2 (C_{r0} V_1 + M_0 Q_1 + Q_0 M_1 + V_0 C_{r1}). \quad \text{Q.E.D.}$$

There is another formula paralleling Theorem III-9 involving the total absolute curvature C_a; C_a on a surface is defined by

$$C_a = \int K_a \, dQ$$

where K_a is the absolute curvature of the surface and dQ is the surface area element. From a formula in differential geometry relating the absolute and relative curvatures,

$$K_a = 1 + K_r$$

it is apparent, after multiplying by dQ and integrating, that

$$C_a = Q + C_r \, . \tag{41}$$

Now another theorem can be proved.

Theorem III-10.

$$\int C_a\ (R_{01})\ dK_1 = 8\ \pi^2\ (C_{a0}V_1 + M_0Q_1 + Q_0M_1 + V_0C_{a1}).$$

Proof.

$$\int C_a\ (R_{01})\ dK_1 = \int (C_r(R_{01}) + Q_{01})\ dK_1$$

by (41).

$$= \int C_r(R_{01})\ dK_1 + \int Q_{01}\ dK_1$$

$$= 8\ \pi^2(C_{r0}V_1 + M_0Q_1 + Q_0M_1 + V_0C_{r1})$$

$$+ 8\ \pi^2\ (Q_0V_1 + V_0Q_1)$$

by Theorem III-9 and (22),

$$= 8\pi^2\ ((C_{r0} + Q_0)V_1 + M_0Q_1 + Q_0M_1$$

$$+ V_0\ (C_{r1} + Q_1))$$

$$= 8\ \pi^2\ (C_{a0}V_1 + M_0Q_1 + Q_0M_1 + V_0C_{a1})$$

by (41).

Q.E.D.

In closing this section, it should be mentioned that there is a statement in 2-space analogous to Theorems III-9 and III-10:

$$\int T_{01}\, dK_1 = 2\pi\, (S_1 T_0 + S_0 T_1 + L_0 L_1).$$

For this particular formula, K_0 and K_1 are two domains with dK_1 the kinematic density for K_1; S_i is the Cayley area enclosed by K_i; L_i is the Cayley length of the boundary of K_i; T_i is the total curvature of K_i. This formula also holds in the Euclidean plane.

ELLIPTIC SPACE OF n-DIMENSIONS

In view of the approach made in the last two sections, the natural question arises about extending these ideas to the n-dimensional case. This brief section will only attempt to summarize some of these extended results.

The formulas concerned with the density of a point, line, plane, and in n-space any k-dimensional linear subspace ($0 \leq k < n$) readily lend themselves to extension in view of their methods of derivation and the ideas presented in (I-1) through (I-4). The same can be said of the kinematic density. Statements concerning the intersections of different geometric objects (see Theorems III-5 and III-6 and the comments which follow each) can be generalized to statements about the intersection of linear subspaces with curves and surfaces. The dimensions of these can vary independently.

A specific formula previously proved (Theorem III-8, the basic formula) has been extended to n-space:

$$dF_1 \; dI = \sin^{n-1} \alpha \; dL_0 \; dL_1 \; d\alpha .$$

Here F_0 and F_1 are two intersecting hypersurfaces. The extension of the density of a line element is the density of frames; hence dI, dL_0, dL_1 represent the density of frames on the intersection, on F_0 and on F_1, respectively. As before, α is the angle between the normals to F_0 and F_1.

The final extension to be mentioned here is perhaps the most interesting and important:

$$I_n \int X(R_0 \cap R_1) \; dK_1 = I_2 \; I_3 \; . \; . \; . \; I_n \left\{ I_n(X_0 V_1 + X_1 V_0) + \frac{1}{n} \sum_{h=0}^{n-2} \binom{n}{h+1} \mu_n(R_0) \mu_{n-2-h}(R_1) \right\} .$$

This is Blaschke's fundamental formula for elliptic n-space. It holds in Euclidean n-space as well. For this formula K_0 and K_1 are two regions with smooth boundary hypersurfaces R_0 and R_1, which have volumes V_0 and V_1, respectively. The (m-1)-dimensional area of a unit sphere in E^m is called I_m; thus $O_2 = 2\pi$, $O_3 = 4\pi$, etc. The μ_i's represent different integral invariants which depend on the dimension of the space being considered; in particular, for 3-space, $\mu_0(R_i)$ is the area of the surface R_i, $\mu_1(R_i)$ is the integral of mean curvature. The Euler-Poincaré characteristic is denoted by X. The cases when n = 2, 3 can be considered as special cases of this general formula.

ACKNOWLEDGMENT

The author wishes to extend his sincere thanks to Professor Chen-Jung Hsu and to mention that the derivations of (31) and the corresponding part of Theorem III-8 were communicated by him.

REFERENCES

Blaschke, Wilhelm. *Vorlesungen über Integralgeometrie.* New York: Chelsea Publishing Company, 1942.

Chern, S. S. *Lectures on Integral Geometry.* Notes by H. C. Hsiao for the 1965 Summer Science Seminar, Taipei, Taiwan. (Mimeographed)

Chern, S. S. "On the Kinematic Formula in the Euclidean Space of n-Dimensions." *American Journal of Mathematics,* 74: 227-236, 1952.

Chern, S. S. "On the Kinematic Formula in Integral Geometry," *Journal of Mathematics and Mechanics,* 16: 101-118, 1966.

Coolidge, J. L. *The Elements of Non-Euclidean Geometry.* London: Oxford at the Clarendon Press, 1909.

Guggenheimer, Heinrich W. *Differential Geometry.* New York: McGraw-Hill Book Company. Inc., 1963.

Olmsted, J. H. M. *Advanced Calculus.* New York: Appleton-Century-Crofts, Inc., 1961.

Petkantschin, Bojan. "Integralgeometrie 6: Zusammenhängen zwischen den Dichten der linearen Unterräume in n-dimensionalen Raum," *Hamburg Abhandlungen,* 12: 249-310, 1936.

Santaló, L. A. *Geometria integral en espacios de curvatura constante.* Buenos Aires: Publicaciones de la Comision Nacional de la Energia Atomica, 1952.

Santaló, L. A. *Introduction to Integral Geometry.* Paris: Hermann & Cie., 1953.

Wu, Ta-Jen. "Integralgeometrie 20: Über elliptische Geometrie," *Mathematische Zeitschrift,* 43: 495-521, 1938.

INTEGRAL GEOMETRY IN CAYLEY SPACES

by

ARTHUR HOWARD SIMONSON

B. A., Phillips University, 1964

AN ABSTRACT OF A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Mathematics

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1967

The study of integral geometry in Cayley spaces was initiated by similar studies in Euclidean spaces which in turn grew out of geometrical studies in probability. This paper, while being devoted mainly to the 3-dimensional case, also gives results for 2-space and n-space.

More specifically, densities of linear subsets are studied thoroughly in 2-space and 3-space. From these results, a number of integral formulas are derived in 3-space, among these being the so-called basic formula and the fundamental formula of Blaschke. Most of the discussion here is devoted to materials presented by T. J. Wu. Some proofs are presented which he omits: other proofs are presented in greater detail, and, hopefully, greater clarity.